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Referentiality in Frege's *Grundgesetze*

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In §§28–31 of his *Grundgesetze der Arithmetik*, Frege forwards a demonstration that every correctly formed name of his formal language has a reference. Examination of this demonstration, it is here argued, reveals an incompleteness in a procedure of contextual definition. At the heart of this incompleteness is a difference between Frege's criteria of referentiality and the possession of reference as it is ordinarily conceived. This difference relates to the distinction between objectual and substitutional quantification and Frege's vacillation between the two.

1. Introduction

Frege devotes §§28–31 of the *Grundgesetze der Arithmetik*[1] to demonstrating that every correctly formed name of his formal language has a reference, a denotation. For Frege a correctly formed name is either a proper name or a function name, sentences being a special kind of proper name. Proper names stand for, or denote, objects, while function names stand for functions. The most prominent objects of the *Grundgesetze* ontology are truth values (the True and the False) and what are roughly set-theoretic constructions (courses-of-values). It is the truth values that sentences stand for, and so part of what Frege hopes to show is that each sentence of his notation stands for a truth value. Now it is perhaps a small step from here to a consistency proof for the logic of the *Grundgesetze*: if it could be shown that the reference of each Basic Law is the True, and that the various rules of inference yield only conclusions standing for the True from premisses standing for the True, then the logic would be shown to be consistent.

Of course the logic of the *Grundgesetze*, as Russell discovered in 1902, is inconsistent. Accordingly, the consistency proof envisioned cannot be carried out. This suggests that Frege's demonstration of the referentiality of all correctly formed names might be incorrect or incomplete, and that even this goal is unattainable. Close examination of Frege's construction, I think, shows an incompleteness in a procedure of contextual definition which is also present elsewhere in the *Grundgesetze*. Frege fails to notice the incompleteness because of the criteria of referentiality he gives (§29), and a gulf which exists between satisfaction of these criteria and referentiality ordinarily or pretheoretically conceived. This gulf forces Frege to oscillate between objectual and substitutional quantification depending on whether he is

1 Georg Ohms Verlagsbuchhandlung, Hildsheim, 1966. Originally published in 1893 (Volume I) and 1903 (Volume II) by H. Pohle at Jena. All references and quotations are from the English translation by Montgomery Furth, *The basic laws of arithmetic* (University of California Press, Berkeley, 1964).

thinking of it in terms of ordinary reference or in terms of his criteria. It is important to recognize that this gulf exists and results in two senses of referentiality.

2. Syntax and semantics

The division of correctly formed names into proper names and function names is a syntactic division; that is, in terms of this division syntactic relationships and properties—like correct formation—are specified. Function names are further distinguished according to numbers of argument positions and the syntactic type of expression appropriate to each position. A function name which takes in its argument places proper names is first-level; a function name which takes in its argument places first-level function names is second-level; and so on. In terms of these parameters every correctly formed name can be assigned a type, and then argument places of function names can also be assigned types depending on the type of expression appropriate to the argument place (§23). For example, proper names are type *1*, first-level functions names of one argument position are type *2*, and first-level function names of two argument positions are type *3*. Now a second-level function name of one argument position can be correctly completed with a type *2* expression; then, its argument position would be of type *2*, while the function name itself would be assigned type *4*. Similarly, if the argument position is correctly filled by a type *3* expression, then the argument place would be type *3*, and the function name would be type *5*. We summarize the hierarchy in Table 1.[2]

Table 1
Hierarchy of Expression Types

Kind of expression	Expression type	Type(s) of the expression's argument place(s)
Proper names	*1*	There are no argument places
Level 1, 1 argument place	*2*	*1*
Level 1, 2 argument places	*3*	*1,1*
Level 2, 1 argument place	*4*	*2*
Level 2, 1 argument place	*5*	*3*
Level 1, 3 argument places	*6*	*1,1,1*
Level 2, 1 argument place	*7*	*6*
Level 2, 2 argument places	*8*	*2,2,*
Level 2, 2 argument places	*9*	*2,3*
Level 2, 2 argument places	*10*	*2,6*
Level 2, 2 argument places	*11*	*3,3*
.	.	.
.	.	.
.	.	.
Level 2, 2 argument places	*16*	*6,3*
Level 3, 1 argument place	*17*	*4*
.	.	.
.	.	.
.	.	.

2 A complete table would also include unequal-leveled function names; see §22.

The criteria of referentiality Frege gives depend on an expression's type:

(A) A first-level function name of one argument place (type *2* expression) has reference if the completion of it with any referring proper name (type *1* expression) is a referring proper name (type *1* expression).
(B) A proper name (type *1* expression) has reference if the completion of any referring first-level function name of one argument place (type *2* expression) with it is a referring proper name (type *1* expression), and if the filling of either argument place of any referring first-level function name of two argument places (type *3* expression) with it is a referring first-level function name of one argument place (type *2* expression).
(C) A first-level function name of two argument places (type *3* expression) has reference if the filling of both argument places by referring proper names (type *1* expressions) is a referring proper name (type *1* expression).
(D) A second-level function name of one argument place whose argument place is of type *2* (a type *4* expression) has reference if the completion of it with a referring first-level function name of one argument place (type *2* expression) is a referring proper name (type *1* expression).
(E) A third-level function name of one argument place whose argument place is of type *4* (a type *17* expression) has reference if the completion of it with a referring second-level function name of one argument place whose argument place is of type *2* (a type *4* expression) is a referring proper name (type *1* expression).

In theory these criteria could continue indefinitely; in practice these five clauses will be enough. The criteria give the impression of a kind of induction: by starting low in the hierarchy with expressions known to have reference, one can show that expressions of greater and greater type also have reference. One problem with this is the establishment of the base clause of the induction: how do we get those expressions initially known to have reference? As we shall see, Frege relies on his informal explanation of the *Grundgesetze* formalism to start the induction off, and this is where some troubles arise.

Any correctly formed compound name is ultimately the result of assembling primitive, or simple, names according to the syntactic laws of composition. For Frege this means that a primitive function name has its argument places filled with the appropriate types of expressions. The filling expressions are, in turn, either primitive, or compounded from primitive expressions according to syntactic laws. And so on back to the simplest compounds of primitive expressions. Now if all of the primitive expressions have reference (that is, satisfy the appropriate criteria from (A)-(E), then the simplest compounds of these will also have reference, for this is precisely what satisfaction of the criteria guarantees. And, again, the simplest compounds of these simplest compounds will have reference. And so on. Consequently, all correctly formed compound names will have reference if the primitive names they are composed of do. Accordingly, Frege sets out to show that the primitive names are referential.

There are no simple proper names in the *Grundgesetze* formalism. There are eight simple function names, five of which are first-level. They are, first, $\zeta = \xi$, the sign for identity. Frege informally explains this by saying that an identity stands for the True if the proper names placed in its two argument places stand for the same object; otherwise it will stand for the False (§7). Secondly, there is '$\text{———}\xi$', the 'horizontal'. Informally, this works much like the predicate 'ξ is the True': when its argument placed is filled by a name of the True, the result stands for the True; when it is completed with any other proper name the result stands for the False (§5). Thirdly, the negation stroke, '$-\!\!\top\!\!-\xi$' simply reverses the value gotten from applying the horizontal (§6). Fourthly, there is the sign for truth functional conditionality with antecedent ξ and consequent ξ: '$\begin{array}{l}-\!\!\top\!\!-\xi \\ \quad\llcorner\!-\zeta\end{array}$'. This stands for false when 'ζ' is replaced with a name of the True and 'ξ' is replaced by a name of the False, or by a name of any other object, and it will result in a name of the True under any other completions by proper names (§12). And, finally, there is '$\backslash\xi$', which in some respects takes the place of the definite article (§11).

There are in addition two simple second-level function names. Frege's notation for universal quantification is '$-\!\overset{\mathfrak{a}}{\smile}\!-\varphi(\mathfrak{a})$'. This will be completed by a first level function name of one argument place (a type *2* expression) to yield a proper name—'$-\!\overset{\mathfrak{a}}{\smile}\!-\Phi(\mathfrak{a})$'—which stands for the True 'if the value of the function $\Phi(\xi)$ is the True for every argument', and stands for the False otherwise (§8). (There is also the third-level function name representing second-order quantification over functions; it will be ignored temporarily.) And, secondly, there is the notation for the course-of-values of a function, '$\overset{,}{\in}\ \varphi(\in)$'. Syntacticaly this works just like quantification, applying to a type *2* expression to yield a proper name, viz. '$\overset{,}{\in}\ \Phi(\in)$'. Rather than give an explicit informal characterization of courses-of values, Frege sketches a contextual explanation which has two parts:

(1) '$\overset{,}{\in}\ \Phi(\in) = \overset{,}{\in}\ \Psi(\in)$' has the same reference as '$-\!\overset{\mathfrak{a}}{\smile}\!-\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$'

and

(2) '$\overset{,}{\in}(\vdash\!\text{———}\in)$' stands for the True,

'$\overset{,}{\in}\ (\in = (-\!\!\top\!\overset{\mathfrak{a}}{\smile}\!-\ \mathfrak{a} = \mathfrak{a}))$' stands for the False.

The first of these stipulations is the infamous Basic Law V from which Russell derived a contradiction. The stipulations of (2) are designed to bridge the gap between courses-of-values and truth values; in particular, they are to enable the evaluation of an identity between what is prima facie a course-of values and what is prima facie a truth value. With these conditions, says Frege, 'we have determined the courses-of-values so far as is here possible' (§10). We will see that these stipulations do not provide for the eliminability of courses-of-values abstracts from all sentential contexts, however, and so as a contextual definition they are inadequate.

3. Referentiality

It follows from the informal characterizations of the horizontal and of identity that $\text{———}\xi$ is the same function as $\xi = (\xi = \xi)$; consequently, the horizontal will have reference if the identity sign does and so no independent check of the referentiality of the horizontal is needed. Since $-\!\!\!\top\!\!-\xi$ has the opposite truth value from $\_\_\_\_\xi$, the negation stroke will have reference if the horizontal does, and so it need not be put to an independent check. And since $\begin{array}{l}\!-\!\!\!\top\!\!-\xi\\ \;\llcorner\!\!-\zeta\end{array}$ is a truth function of two horizontals, it too will have reference if the horizontal does and so need not be independently tested. So of the five primitive first-level function names, only that for identity and '$\backslash\xi$' need be checked for referentiality. These will have reference if filling their argument places with referring proper names results in referring proper names. In the language of the *Grundgesetze* there are two kinds of proper names: those which prima facie stand for truth values, and those which prima facie stand for courses-of-values. By briefly appealing to his informal explanations, Frege argues that the sign for identity and '$\backslash\xi$' result in referring proper names when their argument places are filled with names of truth values; it thus remains to consider completions with courses-of-values names.

Next Frege checks the universal quantifier for referentiality. '$-\!\overset{\mathfrak{a}}{\smile}\!-\varphi(\mathfrak{a})$' will have reference if '$-\!\overset{\mathfrak{a}}{\smile}\!-\Phi(\mathfrak{a})$' has reference whenever '$\Phi(\xi)$' is a referential type 2 expression (by (D)). But if '$\Phi(\xi)$' has reference, then, by criterion (A), '$\Phi(\Delta)$' must have reference whenever 'Δ is a referential proper name. Now, says Frege, 'If this is the case, then this denotation [$\Phi(\Delta)$] either always is the True (whatever 'Δ' denotes), or not always. In the first case '$-\!\overset{\mathfrak{a}}{\smile}\!-\Phi(\mathfrak{a})$' denotes the True, in the second the False' (p. 88).

It is interesting that the reference of a universally quantified sentence is here specified in terms of the references of its instantiation instances. The truth conditions given here are substitutional rather than those of objectual quantification earlier introduced (§8).[3] This shift is forced on Frege by the character of his criteria of referentiality. Whereas in earlier discussions he talks objectually about function names standing for functions and about arguments and values of such functions, the criteria (A)-(E) are couched wholly in terms of the referentiality of substitutional completions of function names. Nowhere in (A)-(E) is any mention made of functions and objects. This seems wrong since it apparently separates referentiality from there being a function referred to. I will return to this below.

Supposing quantification checked, it remains to do two things: (i) check the referentiality of first-level function names when they are completed with courses-of-values abstracts, and (ii) check that the second-level courses-of-values name '$\grave{\epsilon}\varphi(\epsilon)$' has reference. To do (ii), that is, to check that '$\grave{\epsilon}\varphi(\epsilon)$' refers, it must be shown that if '$\Phi(\xi)$' is a referential type *2* expressions, then '$\grave{\epsilon}\Phi(\epsilon)$' is referential (by (D)). But since '$\grave{\epsilon}\Phi(\epsilon)$' has the form of a proper name, it will have reference if and only if: if '$\Psi(\xi)$' is a referential type *2* expression, then '$\Psi(\grave{\epsilon}\Phi(\epsilon))$' has reference, and if '$\Psi(\xi,\zeta)$' is a referential type *3* expression, then the filling of either of its argument places with '$\grave{\epsilon}\Phi(\epsilon)$' is a referential type *2* expression (by (B)). Thus to check the referentiality of '$\grave{\epsilon}\varphi(\epsilon)$', it is sufficient to show:

(3) If '$\Phi(\xi)$' and '$\Psi(\xi)$' are referential, then so is '$\Psi(\grave{\epsilon}\Phi(\epsilon))$'.

3 Compare Frege's *Posthumous writings* (edited by H. Hermes, F. Kambartel and F. Kaulbach; University of Chicago Press, Chicago, 1979), 154, 213.

Note that here it is assumed that '$\Phi(\xi)$' is a type *2* expression while '$\Psi(\xi)$' is either a type *2* expression or a type *3* expression; in particular, identity must be among the '$\Psi(\xi)$'s which are checked. Without loss of generality we can confine our attention to '$\Psi(\xi)$'s which are primitive since it is these which will provide the simplest compounds; if this step results in referential names, then successive compoundings will also. Now to accomplish (i)we need to show that if '$\Psi(\xi)$' is a primitive first-level function name (of type *2* or type *3*), then

(4) If '$\acute{\epsilon}\Phi(\epsilon)$' has reference, then so does '$\Psi(\acute{\epsilon}\Phi(\epsilon))$'.

According to Fregean principles, if an expression lacks reference, then any compound containing it will also lack reference even if it is a correctly formed name. Thus if '$\Phi(\xi)$' lacks reference, so too will '$\acute{\epsilon}\Phi(\epsilon)$'; contrapositively, if '$\acute{\epsilon}\Phi(\epsilon)$' has reference, then so must '$\Phi(\xi)$'. Thus (3) and (4) can both be demonstrated by showing

(5) If '$\Phi(\xi)$' has reference, then so does '$\Psi(\acute{\epsilon}\Phi(\epsilon))$',

(where '$\Phi(\xi)$' is any type *2* expression and '$\Psi(\xi)$' is any primitive type *2* expression or primitive type *3* expression) for the antecedent of (5) is not stronger than the antecedents of (3) or of (4). It is (5) that Frege attempts to demonstrate.

Again, of the five primitive first-level function names, only identity and '$\backslash\xi$' need be checked. To check '$/\xi$' it must be shown that if '$\Phi(\xi)$' has reference, then so does '$/(\acute{\epsilon}\Phi(\epsilon))$'. Once again Frege relies on his informal explanations, this time the ones designed to guarantee the truth of Basic Law VI: if there is an object Γ such that $\acute{\epsilon}\Phi(\epsilon)$ is $\acute{\epsilon}(\epsilon = \Gamma)$, then $/(\acute{\epsilon}\Phi(\epsilon))$ is Γ; otherwise $\backslash(\acute{\epsilon}\Phi(\epsilon))$ is $\acute{\epsilon}\Phi(\epsilon)$. Such explanations assume that the course-of-values abstract is referential, and so Frege's claim of referentiality here depends on the prior checking of identity, for this is the crucial context for courses-of-values abstracts.

To check identity we need to think of '$\Psi(\xi)$' in (5) as identity. This involves showing that

$$\acute{\epsilon}\Phi(\epsilon) = \xi$$

is a referential type *2* expression if '$\Phi(\xi)$' is referential. This, in turn, requires that we show that

(6) $\acute{\epsilon}\Phi(\epsilon) = \Delta$

is a referential proper name when Δ is replaced by a sentence or other name of a truth value, and also that

(7) $\acute{\epsilon}\Phi(\epsilon) = \acute{\epsilon}\Theta(\epsilon)$

is referential if '$\Theta(\xi)$' is. To accomplish this Frege relies on stipulations (1) and (2) mentioned above (p. 7). Stipulation (2) reduces anything of form (6) to an identity statement of the form of (7). And then stipulation (1) reduces anything of form (7) to a quantificational generalization, and this case has already been handled. Unfortunately, as mentioned above, stipulation (1) is tantamount to Basic Law V, which was subsequently discovered to lead to contradiction. It is later revoked, thus leaving the demonstration incomplete. Frege acknowledges this in a letter to Russell, saying that 'It seems, then, . . . that my explanations in §31 are not sufficient to ensure that my combinations of signs have a meaning in all cases'.[4]

4. Circularity

As mentioned earlier, Frege's failure here dooms the allied attempt at a consistency proof for his logic. One way to view this failure is something like this: As is now clear, Frege's attempt to show the referentiality of all sentences of the *Grundgesetze* language relies on the *Grundgesetze* logic. That is, the checks which are made assume that the primitive names have various properties, and these assumptions are reflected in the Basic Laws of the logic. If the logic is inconsistent (as turns out to be the case), then the primitive names cannot have the assumed properties, and the checks have not been made. Any attempted consistency proof along these lines, then, will already assume that the logic is consistent. This is necessary, since, as Gödel's second theorem shows, any consistency proof for a system as strong as Frege's requires a stronger logic than that of the system being considered. So there is no way for Frege to avoid this kind of circular dependence. Still, Frege's construction might well cause us to wonder about the value of consistency proofs in general, since it shows the possibility of 'proving' an inconsistent system consistent using a stronger metalogic.

A further circularity has been attributed to Frege's demonstration by Charles Parsons and Christian Thiel.[5] Parsons reasons that to show the referentiality of '$\acute{\epsilon}\varphi(\epsilon)$' it must be shown that '$\acute{\epsilon}\Phi(\epsilon)$ has reference for all referential '$\Phi(\xi)$' (by (D)). But to know if '$\Phi(\xi)$' is referential, we need to check if '$\Phi(\Delta)$ refers, for all referential proper names 'Δ' (by (A)). But one such 'Δ' is precisely '$\acute{\epsilon}\Phi(\epsilon)$' if everything goes right. So, concludes Parsons, to know if '$\acute{\epsilon}\Phi(\epsilon)$' is referential we must first know if '$\acute{\epsilon}\Phi(\epsilon)$' is referential. What Parsons's line of reasoning shows, at best, is that a frontal, straightforward approach to Frege's goal will not work. But, of course, this is not the path Frege's own demonstration takes. Consequently Parsons's argument does not show Frege's approach to be deficient. To show a theorem cannot be proved one way is not to show that it is unprovable. And to show that one proof is circular is not to show that all proofs are.

If we could rely on Basic Law V (or stipulation (1)), as Frege planned to do, then we could indeed show that an arbitrary sentence of the *Grundgesetze* language is referential in the sense of meeting the criteria (A)-(E) that Frege propounds. Consider, for example,

(8) $\neg\overset{\mathfrak{A}}{\smile}\!-\ \dot{\epsilon}(\epsilon = \epsilon) = \mathfrak{A}.$

4 Jean van Heijenoort (ed.), *From Frege to Gödel* (Harvard University Press, Cambridge, 1967), 127.

5 'Frege's theory of number', in Max Black (ed.), *Philosophy in America* (Cornell University Press, Ithaca, 1965), 189–191; Christian Thiel, 'Zur Inkonsistenz der Fregeschen Mengenlehre', in his (ed.), *Frege und die moderne Grundlagenforschung* (Anton Hain, Meisenheim an Glan, 1975), 151–158.

Since the universal quantifier is known to refer, (8) will have reference if

$$\grave{\epsilon}(\epsilon = \epsilon) = \xi$$

is a referential first-level function name (type *2* expression) (by (D)); and this will be so if

$$(9) \qquad \grave{\epsilon}(\epsilon = \epsilon) = \Delta$$

has reference whenever 'Δ' does (by (A)). Now 'Δ' might stand for a truth value; in this case, by stipulation (2), (9) will stand for the same thing as either

$$\grave{\epsilon}(\epsilon = \epsilon) = \grave{\epsilon}(\underline{\qquad}\,\epsilon)$$

or

$$\grave{\epsilon}(\epsilon = \epsilon) = \grave{\epsilon}(\epsilon = \;\overset{\mathfrak{a}}{\top\!\!\smile}\!\!-\; \mathfrak{a} = \mathfrak{a}),$$

both of which can be shown to have reference by invoking Basic Law V (stipulation (1)). And if 'Δ' is replaced with a smooth breathing courses-of-values abstract, then (9) can be shown to be referential directly by appeal to Basic Law V (stipulation (1)).

We thus show that (8) is referential by showing that the infinity of sentences of the form of (9) all have reference. The question of the referentiality of the sentences of the form (9) is, in turn, reduced to the question of the referentiality of certain universally quantified sentences, thanks to Basic Law V. In this way the question of the referentiality of sentences like (8) which contain courses-of-values abstracts reduces to the question of the referentiality of certain sentences not containing such abstracts.

When introducing a function name, Frege's usual practice, as we have seen, is to tell us its reference by describing the function it stands for; and the function is described as saying what its value is for each possible argument (see §§5, 6, 7, 8, 11, and 12). This procedure can be viewed as giving an explicit definition of the function name. When dealing with the second-level function name for courses-of-values, however, Frege departs from this strategy, giving instead what might be thought of as a *contextual* definition (§10). That is, instead of saying outright what courses-of-values are, he attempts to explain the meaning of any larger context containing a smooth breathing abstract. He does this by equating such contexts with others already understood which do not contain the abstracts. Stipulations (1) and (2) are the equations he gives for doing this. Frege is not forced into this policy because the courses-of-values abstract is a second-level, type *4* expression. For the universal quantifier is of exactly the same syntactic type and yet Frege gives for it (nearly enough) an explicit informal account.

Presumably Frege wants to tell us something enlightening or explanatory about courses-of-values just as in the *Grundlagen*[6] he tries to say something revealing about

6 Translated by J.L. Austin as *The foundations of arithmetic* (Northwestern University Press, Evanston, 1968); original published in 1884 by W. Koebner at Breslau.

numbers. ('How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them?') In this respect courses-of-values and numbers have a different status from that of truth values, familiarity with which is assumed. Because of his desire, Frege is prevented from giving an explicit informal explanation of the smooth breathing courses-of-values abstract, for this could only take the form of saying: '$\grave{\in}\ \Phi(\in)$' stands for the course-of-values of $\Phi(\xi)$. And this is not suitably explanatory; it does not tell us what object that is, or how to 'recognize it again as the same'. We have, that is, no suitable independent means of specifying courses-of-values. The only alternative, then, seems to be supplying conditions of identity like (1) and (2).

5. Eliminability

Notice, however, that stipulations (1) and (2) are not sufficient to provide for the *elimination* of smooth breathing courses-of-values abstracts from all sentences, as one might have supposed from the contextual definition procedure. As a consequence of this failing, truth conditions have not been assigned to all sentences of the *Grundgesetze* notation. There are at least three contexts in which a course-of-values abstract may appear which can not be treated by stipulations (1) and (2). They are: (I) identities as in (8) in which the identity sign is flanked by a smooth breathing abstract on one side and a variable bound by a quantifier on the other; (II) identities as in

$$(10) \quad \smile^{\mathfrak{g}}\!\!-\ [\ \grave{\in}(\in = \grave{\alpha}\ (\alpha = \alpha)) = \grave{\eta}\ g\ (\eta)],$$

in which the identity sign (here, the left-most) is flanked by a smooth breathing abstract on one side and a variable bound by a course-of-values abstract on the other, and (III) contexts such as

$$\smile^{\mathfrak{g}}\!\!-\ [g\ (\ \grave{\in} = \in))\ldots],$$

in which a smooth breathing abstract appears in the argument place of a function variable which is bound by a quantifier.

Contexts (I) and (II) are obtained by employing Frege's second way of forming a name (§30) in which a proper name is excluded from a proper name containing it, thereby forming a new function name. This syntactic operation is seemingly ignored by (A)-(E) as well as in Frege's earlier explanations of the references of his primitive names (§§5*ff*).[7] Context (III) employs the previously ignored higher level quantifier. As is apparent, showing it to be referential involves dealing with courses-of-values and so is not 'similar' (as Frege claims) to lower-level quantification.

One way we might try to treat case (I) is by mimicking the set-theoretic equivalence

$$\{x \mid Fx\} = y \leftrightarrow \forall z\ (z \in y \leftrightarrow Fz)$$

7 Compare Edwin Martin, 'A note on Frege's semantics', *Philosophical studies*, **25** (1974), 441–443.

in Frege's theory using the relation $\xi \frown \zeta$ (§34). Doing so for (8) yields something like

$$\overset{,}{\in}(\in = \in) = y \longleftrightarrow -\!\overset{z}{\smile}\!- [z \frown y = (z = z)].$$

The trouble with this as an elimination is that '$\xi \frown \zeta$' is itself a defined expression, one whose definition includes a clause of the form '$u = \overset{,}{\in} g(\in)$', which is just what we are trying to eliminate. Consequently, expanding our equivalence into primitive terms puts back in the smooth breathing abstract. And were we to count '$\xi \frown \zeta$' as a primitive and undefined relation word, then we would have the further context '$\overset{,}{\in} \Phi(\in) \frown y$' from which to eliminate the course-of-values abstract; and this latter task seems no easier than our original one.

Another way we might try to handle case (I) contexts is by equating sentences like (8) with the infinite conjunction of all sentences of the form of (9); we could then eliminate the abstract by Basic Law V (stipulation (1)). However, this course makes quantification into substitutional quantification, rather than the objectual quantification Frege presumably wants. It also drastically changes the language being considered, now allowing names of infinite length. Finally, we might treat case (I) by invoking two principles Frege holds true: every function has a course-of-values, and every object of the *Grundgesetze* theory is a course-of-values; i.e.,

$$\forall g \, \exists \mathfrak{A} \, (\mathfrak{A} = \overset{,}{\in} g(\in))$$

and

$$\forall \mathfrak{A} \, \exists g \, (\mathfrak{A} = \overset{,}{\in} g(\in)).$$

Given these it seems reasonable that we could replace any expression of the form

$$-\!\overset{\mathfrak{A}}{\smile}\!- \ldots \mathfrak{A} \ldots$$

by the equivalent expression

$$-\!\overset{g}{\smile}\!- \ldots \overset{,}{\in} g(\in) \ldots$$

(8) may thus be converted into

$$-\!\overset{g}{\smile}\!- \quad \overset{,}{\in}(\in = \in) = \overset{,}{\in} g(\in),$$

and thence by Basic Law V into

$$-\!\overset{g}{\smile}\!-\!\overset{\mathfrak{A}}{\smile}\!- (\mathfrak{A} = \mathfrak{A}) = g(\mathfrak{A}),$$

which is devoid of smooth breathing abstracts. But some applications of this procedure may turn a case (I) problem into a case (II) or case (III) problem. For example,

$$-\!\!\overset{\mathfrak{A}}{\smile}\!\!-[\acute{\epsilon}(\epsilon = \mathfrak{A}) = (\mathfrak{A} = \acute{\eta}(\eta = \eta))]$$

will give way to

$$-\!\!\overset{g}{\smile}\!\!-[\acute{\epsilon} = \acute{\alpha}\, g(\alpha)) = (\acute{\alpha}\, g(\alpha) = \acute{\eta}(\eta = \eta))],$$

which contains a type (II) abstract. And, similarly,

$$-\!\!\overset{\mathfrak{A}}{\smile}\!\!-\!\!\overset{g}{\smile}\!\!-[g(\mathfrak{A}) = (\mathfrak{A} = \acute{\epsilon}(\epsilon = \epsilon))]$$

will be replaced by

$$-\!\!\overset{f}{\smile}\!\!-\!\!\overset{g}{\smile}\!\!-[g(\acute{\alpha}\, f(\alpha)) = (\acute{\alpha}\, f(\alpha) = \acute{\epsilon}(\epsilon = \epsilon))],$$

which contains a type (II) appearance of a smooth breathing abstract. Consequently, this solution to case (I) problems must rely on solutions to cases (II) and (III).

Also, a case (II) problem may turn into a case (I) problem as when we use Basic Law V to turn (10) into

$$-\!\!\overset{g}{\smile}\!\!-\!\!\overset{\mathfrak{A}}{\smile}\!\!-[\mathfrak{A} = \acute{\alpha}(\alpha = \alpha) = g(\alpha)],$$

and thence into a case (III) problem when we reduce this in turn to

$$-\!\!\overset{g}{\smile}\!\!-\!\!\overset{f}{\smile}\!\!-[\acute{\epsilon} f(\epsilon) = \acute{\alpha}(\alpha = \alpha) = g(\acute{\epsilon} f(\epsilon))].$$

Consequently, elimination depends on a workable solution to case (III) contexts. All that seems possible here is elimination of the bound function variable by consideration of its instantiation instances, numerous though they are. This route to elimination will, as before, make the language infinitistic. Furthermore, some perfectly good first-level function names, like

$$-\!\!\overset{g}{\smile}\!\!-\ g(\xi) = g(\xi)$$

contain the very same, type (III), context we are trying to eliminate. Thus examination of instances will not ultimately lead to the elimination we are after.

What seems to be the case, then, is that there are at least three contexts containing course-of-values abstracts that we are powerless to explain in already understood terms not containing such abstracts. Stipulations (1) and (2) doe not provide sufficient means to eliminate all abstracts, nor does a satisfactory emendation seem possible. As a consequence, we are left without an explanation of the meanings of some of the sentences of the *Grundgesetze* language. Frege thought that as a result of his showing the referentiality of every *Grundgesetze* name, it had been shown that every sentence 'expresses a sense, a thought. Namely, by our stipulations it is determined under what conditions the name denotes the True' (pp. 89*f*). Of course, since the proof ultimately fails, this latter contention must be withdrawn, and this is what Frege does in his letter to Russell. But the present point is stronger: even had the proof succeeded, even were the *Grundgesetze* logic consistent, still total elimination of courses-of-values abstracts would not be possible. So even allowing Basic Law V, the truth conditions of some sentences are not determined. Consider sentence (10) once again: under what conditions does it stand for the True? We cannot say, for we have been given no general explanation of what courses-of-values are.

6. Context and reference

Frege's criteria (A)-(E) are residue from his earlier *Grundlagen* thesis that it is only in the context of a sentence that we should ask after a word's meaning.[8] This principle suggests the legitimacy of contextual definition, i.e., that a word's meaning can be specified by specifying the meanings of all containing sentences. Now identity provides a very important kind of containing sentence, both because of the primacy of identity as a primitive function name in the *Grundgesetze*, and because of Frege's concern with providing identity conditions. If we want to introduce a type *4* expression, $\hat{x}\,\varphi\,(x)$, then, two important sentence types for which meanings must be given will be

(11) $\hat{x}\,\Phi\,(x) = \hat{x}\,\Psi\,(x)$

and

(12) $\hat{x}\,\Phi\,(x) = \Delta.$

The strategy which Frege adopts with sentences of the form of (11), both in the case of numbers and of courses-of-values, is to equate them with some further sentence

(13) $\tilde{x}(\Phi\,(x), \Psi\,(x))$

involving an already understood type *8* expression. In the case of courses-of-values it is Basic Law V (stipulation (1)) which plays the role of (13). However, this still leaves identities of the form of (12) to deal with, and here two obvious alternatives present themselves:

Alternative 1: Identify $\hat{x}\Phi\,(x)$ with something handy, something already well understood.

Alternative 2: Identify all of the Δs with some $\hat{x}\Phi\,(x)$.

Alternative 1 is the course Frege adopts for numbers, identifying the number of *Fs* ($Nx\,F(x)$) with a course-of-values. This is tantamount to giving an explicit definition of the new function name. It reduces questions about meaning for sentences like (11)

8 Compare Michael Dummett, 'Frege, Gottlob', in *Encyclopedia of philosophy* (ed. Paul Edwards: Macmillan, New York, 1967), vol. 2, 233–234.

and (12) to questions about the meanings of identities involving courses-of-values abstracts. Alternative 1 thus renders superfluous the prior treatment of sentences of the form (11) as sentences like (13). When the definiendum and definiens have matching bound variables (as they do in Frege's treatments), eliminability from all contexts is guaranteed.

Now if Frege is to adopts Alternative 1 also in the case of courses-of-values, he will have to have in hand some objects with which courses-of-values can be identified. For Frege there is only one kind of object which is so basic: truth values. But courses-of-values cannot be identified with truth values because there are very many courses-of-values and only two truth values. So here Alternative 2 must be taken. This requires that all proper names which are not course-of-values abstracts be equated with some name of that form. Then sentences of form (12) will reduce to sentences of form (11), which reduce to sentences of form (13). Since the only names of this sort in the *Grundgesetze* are names of truth values, it is precisely the stipulations of (2) which accomplish this goal. As we have seen, as a contextual definition this fails because there are further sentential contexts made up of identity and quantificational resources from which the courses-of-values abstract is ineliminable. Frege must have at least dimly recognized this ineliminability, for otherwise he should have been prompted to think of the course-of-values abstract (like the one for numbers) as a defined symbol rather than primitive, and to think of Basic Law V as part of a definition rather than an axiom.

Notice the situation Frege is in with respect to courses-of-values. If we allow Basic Law V, then his referentiality proof succeeds. That is, he can show that every name in the *Grundgesetze* meets conditions (A)-(E); and this is what referentiality has been construed to be. Yet even allowing Basic Law V, truth conditions for some of these sentences cannot be given. Under Frege's criteria it is possible to show that a sentence is referential without being able to state its truth conditions. This is just the situation with (10). As a different example, consider a first-level function name '$\Phi(\xi)$'. Suppose we are told the reference of every proper name of the form '$\Phi(\Delta)$' where 'Δ' is a referential proper name; so, then, we know (by (A)) that '$\Phi(\xi)$' is referential. Still, we do not know what function '$\Phi(\xi)$' stands for (if any), but only the values of that function for nameable objects as arguments. Consequently we cannot state truth conditions for '$\smile\!\!\!\!\!-^{\mathfrak{a}}- \Phi(\mathfrak{a})$' in terms of the function '$\Phi(\xi)$' stands for, because we do not know what function that is. When Frege argues for the referentiality of the universal quantifier, therefore, he must do so in terms of the completions of '$\Phi(\xi)$' rather than the function '$\Phi(\xi)$' stands for. Consequently what he can only end up showing is that the substitutional quantifier has reference. So although truth conditions in this case are stated, they are the wrong ones.

We might contrast the two conditions under consideration for the case of a first-level function name, a type *2* expression. On the one hand, it might satisfy criterion (A), and on the other hand it might, intuitively speaking, bear a relation of reference to a function. Prima facie, these conditions are very different. Since Fregean functions are defined for all objects as arguments,[9] satisfaction of criterion (A) is

9 See P.T. Geach and Max Black (eds.), *Translations from the philosophical writings of Gottlob Frege* (Blackwell, Oxford, 1966), 33–34, 165–166.

insufficient to guarantee there is a function referred to. For it might be that although all completions of a function name with proper names result in referential proper names, still there is no appropriate function whose domain is all objects, the unnamed as well as the named. The function has commerce with all objects, while the completions encompass only those objects with names.

We must draw a distinction here if we are to maintain a sense in which 'Everything has a name' is false. Still, a comprehensible and in some ways attractive alternative in the spirit of much of what Frege says is to think of function names as standing for *partial* functions—functions not defined for every object as argument. If we do this, then it is possible for a function name to stand for a (partial) function, but yet not satisfy criterion (A). This will happen if the partial function is undefined for some nameable arguments. People who hold that sentences like '2 is red' are meaningless might well maintain this view: 'ξ is red' stands for a partial function which is undefined for the number 2 as argument. So here there would be a function—a partial function—for which the function name stands although the function name does not satisfy criterion (A). There seems, then, to be a big difference between the two conditions, and this is the gulf earlier mentioned. We have, accordingly, what we can think of as two senses of referentiality, the substitutional sense in which criteria (A)-(E) are satisfied, and the objectual sense in which an expression bears a certain semantic relation to what is usually an extralinguistic entity.

In spite of this difference between substitutional and objectual reference, some people have thought Frege's criteria important. For example, Furth seizes them in an attempt to extricate Frege from his problems with 'the concept *horse*'.[10] But substitutional reference, as a species of reference, is far fetched on the face of it. For, as Quine is fond of reminding us, any syntactical class of expression can be thought of as referential in this sense.[11] We might, for example, isolate a syntactic class (type *p* expressions) which includes 'under', 'above', 'beside', 'in', 'on', 'near', etc. A condition of reference (parallel to (A)-(E)) for an expression of a complementary syntactic type—for example, 'John hit Joe ξ the mouth'—would be that every completion of it with a referential type *p* expression results in a referential sentence. It seems extremely implausible to say on the basis of this that the containing expression stands for some reference, whether it be a function, an object, or something else, for in this case there is no reasonable candidate. Substitutional reference is not objectual reference. If we think—as Frege sometimes seems to—that function names objectually refer to functions, then the problems of attributing or specifying such reference cannot be solved in substitutional terms. Rather, an objectual means of referring to functions must be found.[12]

10 'Two types of denotation', in *Studies in logical theory* (American Philosophical Quarterly Monograph Series, Monograph No. 2; Blackwell, Oxford, 1968), 27–40.

11 For example, *Ontological relativity* (Columbia University Press, New York, 1969), 105–106.

12 Compare Edwin Martin, 'Frege's problems with "the concept *Horse*"', *Critica*, 5, No. 15 (1971), pp. 45–61; and Michael Dummett, *Frege: philosophy of language* (Harper & Row, New York, 1973), 212–217.